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On equations in finite groups and invariants of representations for their subgroups

S.P. Strunkov

Moscow Engineering and Physics Institute, Kashirskoe Shosse 31, 115409 Moscow, Russia Federation

Abstract

Connections between invariants of subgroups of a finite group in a given representation of this group are studied. Also the problem of existence and the number of p-blocks of a finite group are solved in the theory of equations on groups.

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The main aim of this paper is to show connections between some questions in the representation theory of finite groups and in the theory of equations on groups. Some results were published in [8–18].

1. On p-blocks of defect 0

Let G be a finite group, let p be a prime dividing the order $|G|$ of G, and let χ be the character of some complex irreducible representation of G. If $|G| = p^c m$, $\chi(1) = p^{c_1} m_1$ (m and m_1 are prime to p), then the nonnegative number $c - c_1$ is called the p-defect of χ. The following problems of Brauer [1] are very important in the representation theory of finite groups:

(1) What are necessary and sufficient conditions for the existence of characters of p-defect 0?

(2) How many are there?

There is a one-to-one correspondence between characters of p-defect 0 in G and its p-blocks of defect 0; therefore, these questions are equivalent to questions about the existence and the number of p-blocks of defect 0.

More general problems about necessary and sufficient conditions for the existence of p-blocks with a given defect group D and the number of such blocks are reduced via the Brauer correspondence [2] to these problems (1), (2), but for a smaller group; thus, in a sense they are easier than the questions about characters of p-defect 0.

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These problems were first posed 30 years ago; however, it is only recently that active investigation on them has been carried out. One of Tsushima's theorems [19] about idempotents of p-blocks of defect 0 was reformulated in the book of Karpilovski [5] as a wonderful criterion for the existence of such blocks in terms of the equation $xy = g$ (x, y are p-regular). Robinson expressed the number of p-blocks with a given defect group by means of the rank of the product of two matrices whose entries are the cardinals of certain subsets of G [6]. Zhang Ji-Ping found a criterion for the existence of characters of p-defect 0 in groups with cyclic Sylow p-subgroups [20]. The following result gives an infinite collection of such criteria in terms of equations on groups.

Theorem 1.1 (Strunkov [16]). *Let $f(x_1, \dots, x_k, u_1, \dots, u_\ell)$ be a function on a finite group G and let f be a product of elementary functions $[x_i, x_{i+1}]$ and $u_j^{x_s}$ ($x_i \in G, u_j \in G_p$, G_p is a Sylow p-subgroup in G). Moreover, variables, that enter in different elementary factors, are distinct and $k \geq 2$. Then G has a p-block of defect 0 if and only if for some $g \in G$ the number of solutions of the equation $f(x_1, \dots, x_k, u_1, \dots, u_\ell) = g$ does not divide by $p|G_p|^\ell$.*

Corollary 1. *The following conditions are equivalent:*

(1) *A finite group G has a p-block of defect 0.*

(2) *The number of solutions of the equation $[x, y] = g$ is prime to p for some $g \in G$ ($x, y \in G$).*

(3) *The number of solutions of the equation $u^x v^y = g$ does not divide by $p|G_p|^2$ for some $g \in G$ ($x, y \in G, u, v \in G_p$).*

(4) *Let k be a natural number. Then the number of solutions of the equation $[x_1, x_2][x_3, x_4] \dots [x_{2k-1}, x_{2k}] = g$ is prime to p for some $g \in G$ ($x_1, \dots, x_{2k} \in G$).*

It was known before that if G has a p-block of defect 0, then G has two Sylow p-subgroups U, V with $U \cap V = 1$. This fact is a special case of the well-known theorem of Green [4] about defect groups of p-blocks. Obviously, it became a reason for the questions of Brauer. Therefore, it is necessary to get such criteria in structural terms, that is in terms of subgroups, cosets and their intersections. These criteria must make the theorem of Green more precise. Theorem 1.1 gives such a possibility. The equivalence of some conditions of this theorem can be restated in structural terms.

Definition. An ordered pair of Sylow p-subgroups U, V is called a (g, p)-vector ($g \in G$), if $U \cap V = 1$, $g \in UV$ (or in other words, $Ug \cap V \neq \emptyset$).

We note that a (g, p)-vector is a structural object.

Corollary 2. *G has a p-block of defect 0 if and only if the number of (g, p)-vectors in G is prime to p for some $g \in G$.*

Now we consider the question about the number of p-blocks of defect 0 in finite groups in terms of equations on groups.

Theorem 1.2 (Strunkov [16]). *Let s_i be the number of solutions of the equation $[x_1, x_2][x_3, x_4] \dots [x_{2i+1}, x_{2i+2}] = 1$, $\sigma_i = s_i/|G|$,*

$$v_n = \sum \frac{(-1)^{i_1+i_2+\dots+i_n}}{1^{i_1}2^{i_2}\dots n^{i_n} i_1! i_2! \dots i_n!} \sigma_1^{i_1} \sigma_2^{i_2} \dots \sigma_n^{i_n}$$

($n = 1, 2, \dots, r; r = \dim Z(\mathbb{C}G)$; the summation extends over all sets $i_1, \dots, i_n$ of non-negative integers such that $i_1 + 2i_2 + \dots + ni_n = n$).

Then

(1) *all numbers v_n are integers;*

(2) *the number k_0 of p-blocks of defect 0 is equal to the largest n such that the number v_n is prime to p.*

2. On invariants of subgroups of a finite group in a given representation of the group

Now let G be a finite group as before, let H be a subgroup of G, and let χ be the character of an complex representation R of G (now R may be reducible). We consider the number $c_\chi(H) = \sum_{g \in H} \chi(g)$. This number has an interpretation in invariant theory because it is equal to the dimension of the space of fixed points by acting H in R or, in other words, to the dimension of the linear space of invariants of the first degree for H in the restriction of R to H. Now we are interested in relations between the numbers $c_\chi(H)$ for different subgroups H of G. This approach is fruitful. It gives the possibility to generalize some arithmetical facts and to get new applications of relations for these numbers to the theory of finite groups.

We denote by $M(\chi)$ the set of natural numbers, each number a of which is relatively prime to $|G|$ and is representable in the form $p_1^{k_1} \dots p_s^{k_s}$, in which each factor $p_i^{k_i}$ is equal to the order of a finite field in which the representation R can be realized under its reduction to a field of prime characteristic p_i.

Theorem 2.1 (Strunkov [15]). *For any $a \in M(\chi)$,*

$$\sum_{T \supseteq H} \mu(H, T) a^{c_\chi(T)} \equiv 0 \pmod{|N_G(H)/H|},$$

where $\mu(H, T)$ is the Mobius function of the set of all subgroups of G partially ordered by inclusion, and the sum is taken over all subgroups $T \supseteq H$.

If $G = \mathbb{Z}_m$, $H = 1$, χ is the sum of all faithful irreducible characters of G, then this relation turns into Euler's theorem $a^{\varphi(m)} \equiv 1 \pmod m$.

Theorem 2.1 was proved in an algebraic way. Other relations for the numbers $c_\chi(H)$ can be got in a geometric way. In this way, Scott found the following remarkable relation:

$$\sum_{i=1}^{n} c_\chi(\langle a_i \rangle) + c_\chi(\langle a_1 a_2 \ldots a_n \rangle) \le (n-1)\chi(1)$$

for any finite group $G = \langle a_1, \ldots, a_n \rangle$ and its character χ with $(\chi, 1) = 0$ [7]. This relation generalizes the well-known 'Brauer trick' and now it plays a key role in investigations of generators of simple finite groups. The following relation can also be proved in a geometric way.

Theorem 2.2 (Strunkov [17]). *If $G = \langle a_1, \ldots, a_n \rangle$, a_i are involutions and χ does not contain one-dimensional components, then*

$$\sum_{i=1}^{n} c_\chi(\langle a_i, a_{i+1} \rangle) + \chi(1) \le \sum_{i=1}^{n} c_\chi(\langle a_i \rangle),$$

where $a_{n+1} = a_1$.

This relation is similar but not equivalent to Scott's one. It was proved by calculating the first homology group of a two-dimensional compact surface, on which the group G acts. There is a reason for an interest in the numbers $c_\chi(H)$ for small subgroups H. It is in the following fact.

Theorem 2.3 (Strunkov [15]). *For any finite group G and its noncyclic subgroup H,*

$$\sum_{T \subseteq H} (|T|/|N_G(H)|) c_\chi(T) \mu(T, H) = 0.$$

Thus, the numbers $c_\chi(H)$ for noncyclic subgroups H in a sense are defined by the numbers $c_\chi(T)$ for cyclic subgroups T.

Now we want to demonstrate an application of the numbers $c_\chi(H)$ to problems about the possibility to define a group, that is to problems of Burnside type.

Theorem 2.4. *Let n be a natural number, and let $\mathscr{A}$ be a set of nonisomorphic finite groups such that*

(1) *any $G \in \mathscr{A}$ can be generated by not greater than n elements;*

(2) $\exp(G) \le n$ *for any $G \in \mathscr{A}$;*

(3) *every $G \in \mathscr{A}$ has subgroups $H_1, \ldots, H_k$ $(k \le n, |H_i| \le n)$ such that for any $\chi_j \in \mathrm{Irr}(G)$ and for some subgroup H_{s_j} at least one of the inequalities $c_{\chi_j}(H_{s_j}) \le n$ or $c_{\chi_j}(H_{s_j}) \ge \chi(1) - n$ holds.*

Then the set $\mathscr{A}$ is finite.

If we can prove that (3) is a consequence of (1) and (2), then we can prove the restricted Burnside problem for all finite groups (not only for p-groups).

Finally, we demonstrate an arithmetical relation between the numbers $c_\chi(H)$ and some equations on finite groups.

Theorem 2.5. *Let G be a finite group, H its subgroup, $\chi_1, \ldots, \chi_r$ all complex irreducible characters of G, s_k the number of solutions of the equation $u_1^{x_1} u_2^{x_2} \ldots u_k^{x_k} = 1$ $(x_i \in G, u_i \in H)$. Then for any $k \geq 1$,*

$$\sum_{i=1}^{r} \left(\frac{|G|}{\chi_i(1)} \right)^{k-1} c_{\chi_i}(H)^k |H|^k \chi_i(1) = s_k.$$

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